Interpretation of the Structure and Evolution of Adjoint-Derived Forecast Sensitivity Gradients

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ABSTRACT
A 36-h adjoint-based forecast sensitivity study of three response functions defined in the lower troposphere—average temperature in an isolated region of the upper Midwest ($R_1$), meridional temperature difference ($R_2$), and average zonal component of the wind ($R_3$)—is conducted with the goal of providing a synoptic and dynamic interpretation of the sensitivity gradient structure and evolution. In addition to calculating and interpreting the sensitivity gradients with respect to basic model variables along the model forecast trajectory, a technique is outlined that allows for the calculation of the sensitivity gradients with respect to variables derivable from the model state vector (including geopotential, relative vorticity, and divergence), and a method for visualizing the sensitivities with respect to the horizontal components of the wind is proposed and demonstrated.

The sensitivity of $R_1$ to all model and derived variables revealed that $R_1$ was controlled by nearly adiabatic processes associated with the addition or generation of temperature perturbations upstream of the region in which $R_1$ was defined. For $R_2$, the sensitivity gradients revealed the well-known influence of confluent horizontal flow and vertical tilting of isentropes to increase the north–south temperature gradient over the region within which $R_2$ was defined. The sensitivity of $R_3$ to the components of the horizontal wind reveals that simply adding or generating an upstream zonal wind perturbation is insufficient to change the zonal wind at 36 h as these wind perturbations upstream of the domain within which $R_3$ is defined are torqued by the Coriolis force as they are advected toward the domain. These results suggest adjoint-derived sensitivities of quasi-conserved response functions may be more easily interpretable than sensitivities calculated for nonconserved response functions.

1. Introduction
As a complement to observations and analyzed datasets, numerical weather prediction (NWP) models have become an important, and in some cases, essential tool for conducting synoptic case studies. Numerical simulations of phenomena using NWP models provide four-dimensional, (nearly) dynamically consistent datasets at potentially high temporal resolution, which allow for the detailed diagnosis of the structure and evolution of a weather system of interest. In addition, NWP models also allow for the conducting of impact studies (i.e., “what if” experiments) in which one or more model control variables (i.e., the initial conditions, boundary conditions, or parameters used in the parameterization of subgrid-scale processes) are modified and the consequences of these modifications on the phenomenon of interest are evaluated by comparing the resulting simulation against a control simulation at some future time or times. A fundamental limitation of an impact study is that there is no good way to know a priori what type of perturbations (in terms of both size and geographical location) to add to the initial conditions, making trial and error an important aspect to such studies. While impact studies typically involve a small number of specific perturbations to the model control variables, they may be used to deduce the effect each such perturbation has on any aspect of the NWP model forecast state.

A sensitivity study, on the other hand, involves evaluating the change in a specific aspect of the model forecast state (called a response function) associated with arbitrary but sufficiently small changes in any of the model control variables at the initial or forecast times. As defined here, a sensitivity study calculates the gradient of the response function with respect to the model control variables (Errico 1997).

A set of model impact studies is a useful, but not
particularly efficient way to evaluate forecast sensitivities. In general, using impact studies to evaluate forecast sensitivities tends to be computationally expensive, because to evaluate the sensitivity to the full initial state vector, one would need to perturb each variable at each grid point, or any combination of grid points, and integrate the NWP model forward. The number of elements composing a model state vector for operational NWP models can typically be on the order of at least $10^6$, making this an impractical means of evaluating forecast sensitivities. A highly efficient means of assessing forecast sensitivity is to perform a single integration of the adjoint of the linearized NWP model to evaluate the sensitivity of a particular forecast aspect to the full initial state vector (Hall and Cacuci 1983; Hall 1986; Errico and Vukčević 1992; Errico 1997).

The calculation of sensitivity gradients using an adjoint model has been applied to a variety of meteorological applications. In data assimilation, the adjoint model can be used along with minimization algorithms to efficiently determine an optimal analysis of a model initial state by combining knowledge from a previous model state and observations, as well as the error characteristics of each (e.g., Talagrand and Courtier 1987). Adjoint models have also been applied to problems concerning parameter estimation and stability analysis (Hall et al. 1982). The computation of optimal perturbations, and in particular singular vectors (Farrell 1989; Buizza et al. 1993), requires an adjoint model. Much like in the aforementioned forecast sensitivity study, assuming a perfect model and some knowledge of actual forecast error, an adjoint model can also be used to evaluate possible analysis errors (Rabier et al. 1992; Klinker et al. 1998; Langland et al. 2002). Sensitivity calculations have also been used in combination with observations to improve analyses and subsequent forecasts, as in Hello et al. (2000). Despite these various applications, few studies exist that provide physical interpretations of sensitivity fields, their evolution, or their relation to the basic state fields from which they were calculated. In this contribution, we hope to provide synoptic and dynamical interpretations of adjoint-derived forecast sensitivity fields and present useful techniques for visualizing these fields. The work that follows represents a culmination of an assessment of real-time forecast sensitivity fields and their evolution computed twice daily over the last 2 yr.\footnote{The real-time forecast sensitivity fields may be viewed online at http://helios.aos.wisc.edu.}

The definition of an adjoint model and a description of its use for sensitivity applications are presented in section 2. This is followed in section 3, by a review of literature on sensitivity studies that provide the motivation for the current work. Section 4 describes in detail the modeling system and data used, the case studied, and the linearity tests conducted. Interpretations of sensitivity gradients with respect to basic model variables are presented in section 5. Section 6 describes the evaluation and interpretation of sensitivities with respect to derived variables respectively. The evaluation of sensitivity gradients for other response functions is provided in section 7. Finally, a summary of our findings as well as evaluation of the method can be found in section 8.

\section{Defining an adjoint model\footnote{A more extensive discussion of adjoint models may be found in Errico (1997).}}

Mathematically, the adjoint of a linear operator $A$, is an operator $A^*$, that satisfies the relation

$$\langle x, Ay \rangle_m = \langle A^*x, y \rangle_m, \quad (1)$$

where $x$ and $y$ are vectors in $R^m$ and $R^n$, respectively, which denote the vector space of $m$- and $n$-dimensional real vectors and $<_m$ and $<_n$, denote inner products in $R^m$ and $R^n$, respectively. This definition of an adjoint operator, coupled with the notion that the adjoint of a linear operator is the transpose of the matrix representation of that operator, provides a powerful, practical means of analytically and numerically generating adjoints for continuous and discrete operators.

The notion of an adjoint model is best motivated by considering its use in performing a model sensitivity study. We first generalize the concept of a model to be any (perhaps nonlinear) operator, $N$, having an input, $x_{in}$, and an output, $x_{out}$ (Fig. 1): $x_{out} = N(x_{in})$. For a sensitivity study, any differentiable function, $R(x_{out})$, of the model output, $x_{out}$, may be used to define a forecast response function.
If the input to the model is perturbed by $\delta x_{in}$, then it can be anticipated that the output of the model will also change by some amount, $\Delta x_{out}$. Provided that the perturbations to the model input, $\delta x_{in}$, are small, we may relate these small changes to changes in the model output, $\Delta x_{out}$.

$$\Delta x_{out} \equiv \delta x_{out} = \frac{\partial N}{\partial x_{out}} \delta x_{in} = L \delta x_{in}$$  \hspace{1cm} (2)

where $L$ is the linearized model operator which maps perturbations of the model input, $\delta x_{in}$, to perturbations of the model output, $\delta x_{out}$ (Fig. 1).

Changes in the model output state imply changes to the value of a response function, $\Delta R = R(x_{out} + \delta x_{out}) - R(x_{out})$. Linearizing this difference allows $\Delta R$ to be approximated by the differential, $\delta R$:

$$\Delta R \equiv \delta R = \left\langle \frac{\partial R}{\partial x_{out}}, \delta x_{out} \right\rangle,$$  \hspace{1cm} (3)

provided that the perturbations to the model output, $\delta x_{out}$, are small. Using (2), and the definition of the adjoint of an operator (1), we may rewrite (3) as

$$\delta R = \left\langle \frac{\partial R}{\partial x_{out}}, \delta x_{out} \right\rangle = \left\langle \frac{\partial R}{\partial x_{out}}, L \delta x_{in} \right\rangle,$$  \hspace{1cm} (4)

Using the definition of a differential,

$$\delta R = \left\langle \frac{\partial R}{\partial x_{in}}, \delta x_{in} \right\rangle,$$  \hspace{1cm} (5)

we may define:

$$\frac{\partial R}{\partial x_{in}} = L^* \frac{\partial R}{\partial x_{out}}.$$  \hspace{1cm} (6)

Therefore, the output of the adjoint model is the gradient of a response function with respect to the model input (i.e., the sensitivity of the response function to model initial conditions), and can be utilized to estimate the change in the value of that response function associated with any arbitrary, but small, perturbation to the model input. Note, that in contrast to the nonlinear and linearized models that relate model input to model output, the adjoint model maps the gradient of the response function with respect to the model output to the gradient of the response function to the model input (Fig. 1). The adjoint of an NWP model is integrated “backwards” in time to obtain the gradient of a response function to the model state at some earlier forecast time.

3. Motivation

Perhaps the most common framework for utilizing forecast sensitivities is in the evaluation of processes associated with cyclogenesis. Some studies have chosen to evaluate forecast sensitivities in an idealized setting (Rabier et al. 1992; Langland et al. 1995). In both studies, the response functions chosen were primarily related to the surface pressure associated with the center of a developing cyclone, as well as lower tropospheric kinetic energy. It was found that cyclogenesis was most sensitive to lower and middle tropospheric temperature and wind perturbations. In particular, the sensitivity gradients were found to exhibit an upshear vertical tilt, implying that cyclogenesis was most sensitive to baroclinic perturbations. The sensitivity gradients seem to contain some notion of balance, as the sensitivities with respect to temperature and wind were related to sensitivity with respect to the potential vorticity (PV) distribution (Langland et al. 1995). Langland et al. (1995) found that sensitivities to surface heat fluxes as well as surface friction for an idealized cyclogenesis event were maximized in the warm sector of the developing cyclone.

A similar methodology has been applied to observed cases of cyclogenesis (e.g., Errico and Vukičević 1992; Vukičević and Raeder 1995). In these studies, several response functions including surface pressure, a weighted distribution of surface pressure, as well as relative vorticity were chosen to evaluate the sensitivity of surface cyclone intensity to model variables. Errico and Vukičević (1992) found that the sensitivity gradients for relatively short lead times exhibited gravity wave-like characteristics for certain response functions that are sensitive to such processes, such as surface pressure, whereas for longer lead times the sensitivity gradients were of larger scale and in general associated with synoptic features such as a middle-tropospheric geopotential trough and upstream ridge. To aid in the interpretation, sensitivity to geopotential height was derived from the distribution of temperature sensitivity. It was also shown that forecast sensitivities can exhibit a strong dependence on the choice of response functions. Vukičević and Raeder (1995) take a somewhat different approach, in which they use sensitivity gradients to formulate perturbations to the analysis, to more clearly determine potential triggers for mountain lee cyclogenesis. Their formulation of PV perturbations from adjoint-derived sensitivities seems to also suggest that the forecast sensitivities may contain some notion of balance.

Adjoint models have also been utilized in the evaluation of sensitivity to initial conditions for estimates of forecast error, and to estimate key analysis errors. Like the studies related to cyclogenesis, sensitivity of forecast error exhibits a characteristic upshear baroclinic tilt, with maxima in the lower and middle troposphere (e.g., Rabier et al. 1996). When forecast error is used as

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3 The reader is also directed to Langland and Errico (1996) and Vukičević and Raeder (1996).
the response function, the initial condition sensitivity can be used to derive estimates of so-called key analysis errors, which when subtracted from an analysis, can lead to significant reductions in the subsequent forecast error (Rabier et al. 1996; Klinker et al. 1998; and Langland et al. 2002).

Despite all of this work in which adjoint sensitivities are applied and qualitatively described, there is a lack of an interpretation of the sensitivity fields themselves. Although different response functions have been used in these studies, most have been related to surface cyclogenesis, and in particular, have been a function of surface pressure or relative vorticity. Sensitivity gradients for other response functions can be drastically different than those calculated for diagnosing cyclogenesis, as was shown in studies where lower tropospheric moisture flux (Lewis et al. 2001) and precipitation rate (Errico et al. 2003) were chosen as response functions. A careful formulation of different response functions can lead to greater insight into physical processes.

Beare et al. (2003), in an impact-derived forecast sensitivity study, suggest a weakness in the dynamical interpretation of adjoint-derived forecast sensitivities is the lack of a time evolution component. Indeed, previous studies have focused primarily on sensitivity to initial conditions, and in a few cases, sensitivities to intermediate model states without a description of the evolution of forecast sensitivity.

In this work, we focus on dynamical and synoptic interpretations of forecast sensitivity gradients and their evolution, as opposed to providing solely a qualitative description of the forecast sensitivities to initial or intermediate model states. We choose to apply the adjoint method to a relatively simple case, using a variety of response functions. The response functions chosen for this work have been carefully formulated in such a way that the forecast sensitivity gradients can be easily interpreted. We hope to provide more insight into the nature of forecast sensitivity by interpreting the evolution of the sensitivity gradients. It was shown in Errico and Vukicevic (1992) that sensitivity to variables other than those contained in the model state vector can be derived, and aid a great deal in the interpretation of the sensitivity gradients. We employ this methodology in our description of the sensitivity gradients by deriving sensitivities to synoptically relevant variables such as relative vorticity and geopotential.

4. Model and case description

a. Modeling system and data

The adjoint model used for this study is a component of the Pennsylvania State University/National Center for Atmospheric Research Fifth Generation mesoscale model (MM5) Adjoint Modeling System (Zou et al.

1997). This modeling system, based upon version one of MM5, includes the nonlinear MM5 model, its TLM, and corresponding adjoint. The MM5 model is a non-hydrostatic, limited area, primitive equation model, which uses as its vertical coordinate a terrain following sigma coordinate. For all of the sensitivity calculations performed, the nonlinear version of MM5 is used to create a basic state about which the TLM and adjoint models are linearized. For the TLM and adjoint integrations, the basic state is updated every time step. The adjoint code has been modified by the first author in a manner as described by Zou et al. (2001) in order to eliminate the nonphysical oscillation in the sensitivity gradients associated with the adjoint of the leap frog time stepping scheme. This modification allows for an evaluation of the time evolution of the forecast sensitivities.

The domain for the nonlinear, TLM, and adjoint integrations is a 90 km, 70 × 48 horizontal grid (model domain is shown in Fig. 3), with 10 evenly spaced sigma levels in the vertical (top pressure level in the model is 100 hPa). The nonlinear model is initialized from the National Centers for Environmental Prediction (NCEP) Eta Model analysis (on the AWIPS 104 grid) interpolated to the MM5 grid, and lateral boundaries are updated using the NCEP Eta Model forecast. The nonlinear integrations use the following physical parameterizations: the Grell convective scheme, a bulk aerodynamic formulation of the planetary boundary layer, horizontal and vertical diffusion, dry convective adjustment, and explicit treatment of cloud water, rain, snow and ice. The TLM and adjoint integrations use the same parameterizations (or their adjoints), but the effects of moisture are neglected. This means that the TLM and adjoint models are integrated using only dry dynamics about the moist basic state created from the nonlinear model run.

b. Case overview

This study focuses on a 36-h MM5 forecast initialized at 1200 UTC 10 April 2003. In particular, we are interested in the forecast over the upper Midwest region, which for this forecast period is dominated by the passage of a baroclinic zone and trough in geopotential height trailing from a lower tropospheric cyclone propagating through southern Canada (Fig. 2). The passage of the trough associated with the Canadian cyclone is consistent with a lower tropospheric wind shift from a southwesterly direction early in the forecast period, to northerly winds later in the forecast period in regions of the upper Midwest. Along with this wind shift, the temperature advection changes from weakly positive or neutral early in the forecast, to significantly negative later in the period, with strong cold advection dominating the entire upper Midwest throughout the final 12 h of the forecast period. Temperatures at 850
hPa decrease by as much as 10°C over Wisconsin for this 36-h forecast.

c. Linearity test

For any adjoint-based sensitivity study, it is necessary to first evaluate the validity of the case dependent tangent-linear assumption (Errico and Vukićević 1992). To verify that the linearity assumption is valid, perturbations that are evolved linearly via the TLM are compared with difference fields obtained from two nonlinear model forecasts. We note that Errico and Vukićević (1992) suggest that the validity of the linear approximation may be dependent on size and type of perturbation, and mention that the effects of discontinuous or highly nonlinear processes can be important because initial perturbations may excite highly nonlinear processes.

A 1° temperature perturbation at a single grid point on the σ = 0.75 surface in central Saskatchewan (Fig. 3a) is used to assess the assumption that the linearized dynamics are relevant for this case. While in general any perturbation could be added to any grid point or grid points, this grid point was chosen because it was identified as the most sensitive grid point for the 36-h forecast of temperature over the upper Midwest (to be discussed later). We compare the 36-h evolution of this temperature perturbation in the TLM with two nearly identical nonlinear, full physics forecasts. The nonlinear forecasts differ only in their initial conditions, where the 1° temperature perturbation is added to the analysis for the second nonlinear model run. If the linear approximation is valid, the linearly evolved perturbation should be the same as the nonlinear difference field valid at the same time.

It is clear for the perturbation we have chosen that diabatic effects are important, as the linearly evolved perturbation (Fig. 3a) looks almost nothing like the nonlinear difference field outside of the upper Midwest (Fig. 3b). This is likely due to an adjustment process in the nonlinear model, which excites gravity wave–like perturbations that eventually trigger the convective scheme over conditionally unstable regions over the Atlantic and Pacific Oceans. This rapid growth of small perturbations due to the triggering of a convective parameterization is similar to that reported by Zhang et al. (2002). These types of perturbations are not evident in the TLM evolution because of the exclusion of diabatic effects, and in particular the convective parameterization. It is important to note that the linearity test seems to hold quite well in the upper Midwest, the region in which we are most interested for this case. Outside the upper Midwest, however, the differences
are problematic since the perturbations that arise as a result of the triggering of the convective scheme are an order of magnitude larger than the linearly evolved perturbations in the upper Midwest.

There is an inconsistency in performing a linearity test in this framework, since the version of the TLM being used has no information of diabatic processes, other than in the model tendencies included in the basic
state. Therefore, a comparison with the nonlinear difference fields is not a relevant comparison, as we are not comparing evolutions using the same model physics. It is inherently difficult to linearize such things as convective parameterizations, due to their nonlinear nature, and in particular, the use of discontinuous switches (e.g., “if–then” statements), which define when and where the cumulus scheme is triggered. As a result, the linearized versions of such parameterizations are sometimes unreliable and often excluded in adjoint-based sensitivity studies (Errico and Vukičević 1992; Langland et al. 1995), making it difficult to perform an “exact” linearity test when including diabatic processes in the nonlinear model.

We can however make a direct comparison by using the same set of physics in our TLM and nonlinear model integrations, which in this study, means turning off the diabatic physics in our nonlinear model integrations. When we recalculate a TLM evolution of the initial temperature perturbation as previously described, but this time about a basic state created from a dry version of the nonlinear model (Fig. 4a), we get a result that compares favorably with a nonlinear difference field calculated from two nonlinear model runs that exclude moist physics (Fig. 4b). For dry physics simulations, the linearity assumption appears to be valid.

There are, however, important differences when we include moist processes when creating the basic state, in terms of the shape and magnitude of the linearly evolved temperature perturbation (Figs. 4a,c). When we evolve the temperature perturbation about a moist basic state, the resultant temperature perturbation (Fig. 4c) more closely resembles the shape and magnitude of the nonlinear difference field that included moist physics (Fig. 4d). We can therefore conclude that we are gaining useful information by performing our linear calculations about a moist basic state.

In summary, when we exclude the effects of moist processes, it is clear that the linear assumption for this case holds quite well. Since we are aware of the problems involved in using the linearized versions of cumulus parameterizations, we choose to exclude the linearized versions of diabatic processes from our TLM and corresponding adjoint model integrations. However, we can include some of the effects of moist processes by integrating the TLM about a basic state that was created from a nonlinear model that included diabatic effects. It is evident that for the portion of the forecast we are most interested in, the linearity assumption appears valid even if we include diabatic processes in our nonlinear difference calculations, because the upper Midwest appears to be far enough removed from those re-

![Fig. 4. As in Fig. 3, except calculated from the TLM integrated about (a) a dry basic state and (c) a moist basic state; as well as from the difference between two nonlinear forecasts that used (b) dry physics and (d) moist physics.](image)
gions in which the cumulus parameterization is very sensitive. It is likely this problem could be avoided all together if we were to use perturbations that would not excite waves through adjustment, and therefore not trigger the convective scheme beyond those regions which were already triggered in the original nonlinear integration.

5. Interpretations of sensitivity fields and evolution

a. Definition of response function

For this study, we choose the 36-h forecast average temperature on the $\sigma = 0.85$ surface in a box in the upper Midwest (Fig. 5d) as the response function (hereafter denoted $R_1$). The input to the adjoint model, the gradient of $R_1$ with respect to the model forecast state at $\tau = 36$ h, is zero everywhere, except with respect to temperature in the box used to define the response function. This gradient serves as the initial condition for the adjoint model’s integration “backwards” in time to evaluate the gradient of the response function with respect to the model state at forecast times $\tau, 0 \leq \tau < 36$ h.

b. Interpretation of sensitivity to temperature

The maximum temperature sensitivity with respect to the model analysis is spatially isolated, located on the cold side of a baroclinic zone, and is characterized by having two local maxima in southern Canada, with regions of much smaller negative temperature sensitivity to the east and west (Fig. 5a). A cross section at the initial time through the most sensitive regions shows that the sensitivity of $R_1$ to temperature exhibits a tilt in the vertical that in general is along the isentropes, with a maximum axis along the 288-K isentropic surface (Fig. 6a). While the largest sensitivity gradients at the initial time are confined to the lower troposphere, the maxima in the sensitivity gradients at this time are located above the level in which the response function is defined at the final time.

Using (5), we see that if we were interested in increasing the amplitude of the response function by perturbing the model initial state, we would have to add positive (negative) perturbations where the sensitivity

![Fig. 5. Evolution of wind (barbs, m s$^{-1}$), temperature (thin dashed, every 3 K), and sensitivity of $R_1$ with respect to temperature (heavy contours, every $1.5 \times 10^{-3}$ K K$^{-1}$, negative contours dashed and zero line omitted), and TLM evolved temperature perturbation (shaded every $0.5 \times 10^{-2}$ K) on the $\sigma = 0.85$ at selected forecast times: (a) 0, (b) 12, (c) 24, and (d) 36 h for forecast initialized at 1200 UTC 10 Apr 2003. Lines A-B in (a) and C-D in (c) indicate the orientation of cross sections for Fig. 6.]
gradients shown in Fig. 5a are positive (negative). For example, adding a positive temperature perturbation to the initial condition over Saskatchewan or Manitoba where the gradient with respect to temperature is positive (Fig. 5a), would lead to an increase in $R_1$. This result can be seen in our linearity tests, where adding a positive temperature perturbation to the analysis in Saskatchewan on the $\sigma = 0.75$ surface, resulted in an increase in temperature (relative to the basic state) over the upper Midwest 36 h later (Fig. 4c), which was concomitantly associated with an increase in $R_1$.

The maximum sensitivity with respect to temperature at $\sigma = 0.85$ remains spatially isolated for all forecast times from $\tau = 0$ h to $\tau = 36$ h and increases in magnitude with increasing time (Fig. 5). The sensitivity gradient being consistently in an environment of northwesterly winds for all forecast times suggests that advection plays a key role in determining the value of this response function. That the initial temperature sensitivity gradient identifies source regions for the air mass that ends up in the box defining the response function at the 36-h forecast time may be verified by considering the final location of parcel trajectories seeded to begin in the most sensitive regions for the 36-h forecast. Because the flow is nearly adiabatic, isentropic parcel trajectories are considered. Trajectories on the $285$-K isentropic surface, which is approximately the surface that intersects $\sigma = 0.85$ over the Midwest at $\tau = 36$ h, do in fact show that parcels that begin in the sensitive regions end up over the region in which the response function is defined (Fig. 7). Furthermore, a cross section taken through the maximum sensitivity at $\tau = 24$ h shows that the maximum positive values are more localized about the $\sigma = 0.85$ surface (Fig. 6b). However, the axis of maximum temperature sensitivity is still along the $288$-K isentropic surface at $\tau = 24$ h, which is also consistent with the notion that adiabatic advection is important for this particular response function.

The evolution of the temperature perturbation used in the linearity tests may be compared with the evolution of temperature sensitivity also shown in Fig. 5. Note that both propagate to the southeast at approximately the same speed. The perturbation propagates just ahead of the southern of the two local maxima in the sensitivity gradient for the 12- to 36-h forecast times, despite the fact that the initial perturbation is located on the $\sigma = 0.75$ surface and in the middle of the

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4 For brevity in this paper, only those perturbations that lead to an increase in a response function are considered. Because of linearity, if perturbations of opposite sign to those discussed in text are instead considered, the effect on the response function would be of the same magnitude but of opposite sign.

5 The initial temperature perturbation in the impact experiment results in a positive nonlinear difference field over the upper Midwest (Fig. 4d). The actual change in the $R_1$ associated with this field is $\Delta R_1 = 0.009$ K. We can also then estimate the change in the value of the response function by taking the product of the 1° perturbation with the value of the gradient with respect to temperature at the grid point, which was perturbed [refer to (5)], which also gives a change in the response function of $\Delta R_1 = 0.009$ K. This is further evidence that the linearity assumption appears to be valid for this case.
northern of the two local maxima in temperature sensitivity. It is important to note that the spreading out of the temperature perturbation from a single point to a finite area indicates that model diffusion as well as advection play important roles in the evolution of both the perturbations as well as the sensitivity fields.

c. Sensitivity to vertical motion and perturbation pressure

Perturbing the temperature distribution is not the only way to change the response function; consequently, an interpretation of forecast sensitivities with respect to other basic model variables is necessary. The horizontal and vertical distributions of the sensitivity gradients with respect to temperature (Figs. 5c and 6b), vertical velocity (Figs. 8a,c), and perturbation pressure (Figs. 8b,d) at 24-h are very similar in scale and structure, although of opposite sign.

The similarity between the sensitivity gradients exists because of the importance of quasi-horizontal temperature advection for $R_1$. For example, adding subsidence at $\tau = 24$ h in northern Wisconsin and Minnesota, where the sensitivity with respect to vertical motion at that time is negative (Fig. 8a), would lead to an increase in the response function at 36-h. This is consistent with adiabatic warming associated with a downward advection of potential temperature (Fig. 8c) in the region identified to be sensitive in terms of temperature (Fig. 5c). This warm temperature perturbation would then be advected downstream, and increase the forecasted temperature in the box.

The gradient with respect to perturbation pressure is best interpreted by examining the cross section in Fig. 8d. Perturbing the perturbation pressure is equivalent to perturbing the full pressure. Consequently, adding negative pressure perturbations in the lower troposphere (below $\sigma = 0.75$) over the western Great Lakes, where the sensitivity with respect to perturbation pressure is negative (Figs. 8b,d), would lead to an increase in the forecast temperature over Wisconsin. At the same time, we would need to add a positive pressure perturbation above the $\sigma = 0.75$ level in the same region, where the perturbation pressure sensitivity is positive (Fig. 8d). This is consistent with an increased thickness, if two isobaric surfaces are considered, one

![Fig. 8. Sensitivity of $R_1$ with respect to vertical velocity (heavy contours, interval $5 \times 10^{-3}$ K m$^{-1}$ s$^{-1}$) (a) at $\sigma = 0.85$ and (c) in a vertical cross section: as well as with respect to pressure perturbation (heavy contours, interval $2 \times 10^{-6}$ K Pa$^{-1}$) (b) at $\sigma = 0.85$ and (d) in a vertical cross section, valid for the 24-h forecast initialized at 1200 UTC 10 Apr 2003. Negative contours are dashed, and the zero line has been omitted. Cross-section orientation for (c) and (d) is shown in Fig. 5c. Cross sections in (c) and (d) also show potential temperature (light dashed, interval 3 K).](image)
above \((p_A)\) and one below \((p_B)\) the level about which we need to perturb in a dipole fashion. To decrease (increase) the pressure below (above), we would need to lower \(p_B\) (raise \(p_A\)) (Fig. 9). An increase in the thickness between two isobaric surfaces is directly proportional to an increase in the column-averaged temperature between those two surfaces, or a generation of a warm temperature perturbation that can subsequently be advected downstream.

d. Sensitivity to horizontal wind

The sensitivity gradients with respect to the horizontal components of the wind can also be linked to the generation of temperature perturbations, despite the fact that their spatial distribution is much different than that of the other sensitivity gradients (Figs. 10a,b). The sensitivities with respect to the components of the horizontal wind consist of dipole structures, as seen in Fig. 10. To compare with the previous description, consider how the wind field needs to be perturbed at 700 hPa at the 12-h forecast time to increase the value of the response function. By interpreting the sensitivity gradients, it is seen that we would need to increase the zonal wind over southeastern Manitoba and decrease it in the Dakotas and Minnesota, where the sensitivity gradients are positive and negative respectively (Fig. 10a). Likewise, we would need to increase the meridional wind in the Dakotas and western Manitoba and decrease it over southwestern Ontario and northern Minnesota (Fig. 10b).

![Figure 9](image)

Fig. 9. Schematic showing interpretation of sensitivities of average temperature to perturbation pressure. Increasing (decreasing) the perturbation pressure in a region of negative (positive) sensitivity results in an increase (decrease) of the height of nearby isobaric surfaces. In the example shown, it is assumed that the upper \((p_A)\) and lower \((p_B)\) isobaric surfaces are flat and separated by a thickness of \(\Delta\) at some time \(t\) before the time at which the response function is defined. Assuming a vertical dipole in the forecast sensitivities gradients with positive sensitivities aloft, it is seen that modifications of the forecast state perturbation pressures at time \(t\) result in an increase in the local column thickness \((\Delta > \Delta_p)\) and hydrostatically, to an increase in the layer mean temperature. This locally warmed column is subsequently advected into the region defining the response function.

Rather than displaying the sensitivities with respect to the horizontal wind components as separate scalar fields, an alternative is proposed: define each of the gradients of a response function with respect to the horizontal wind components as components of a sensitivity vector. Because one is typically interested in perturbing a distribution of wind vectors, rather than a single wind component, the advantage of this representation is that greater insight into the relative magnitudes and directions of the required perturbation to the wind field to get maximum changes in a given response function is afforded. In addition, combining the sensitivities with respect to the horizontal wind components makes it more apparent which of the horizontal wind components is more sensitive.

For the response function \(R_1\), the vector representation of the sensitivity gradients with respect to the horizontal wind is particularly useful for evaluating the impact of wind perturbations added to the initial or forecast model state. Figure 10c indicates that increasing the 12-h forecast wind from the south and southeast over the Dakotas, a direction nearly perpendicular to the basic state isotherms, will lead to an increase in \(R_1\). An increase of winds in this direction relative to the basic state isotherms will generate positive temperature perturbations, which will subsequently be advected downstream. This is interpreted as a decrease in the magnitude of the basic state northwest wind and its associated cold advection. Finally, note that the pattern of sensitivity vectors implies that the addition of a cyclonic (anticyclonic) circulation over southern Saskatchewan (northern Minnesota) at 12-h into the forecast will lead to an increase in \(R_1\) (Fig. 10c).

6. Sensitivity to derived variables

Because the output variables of the MM5 forward model are \(u, v, w, T, p',\) and \(q_c\), the output of the adjoint model is restricted to gradients of a chosen response function with respect to only these variables. It is potentially more insightful to consider sensitivities with respect to derived variables, such as the vertical component of relative vorticity (hereafter referred to as “relative vorticity,” \(\zeta\)), horizontal divergence \((\delta)\), or geopotential \((\phi)\). Because mesoscale and synoptic weather systems may be identified with upper tropospheric shortwaves, midtropospheric vorticity maxima, or divergent, secondary circulations, consideration of sensitivities with respect to \(\zeta, \delta,\) or \(\phi\) allows for an assessment of how changes to these particular synoptic features might change the value of a response function. Thus, we seek an operator that has as its input the sensitivities with respect to the model variables and as its output the sensitivities with respect to a derived variable. This operator must then be the adjoint of an operator that inverts the distribution of a derived variable to yield distributions of the model variables (Fig. 11).
a. Sensitivities with respect to relative vorticity

As an example, consider a calculation of the sensitivity of $R_1$ with respect to vorticity, given the sensitivities of $R_1$ with respect to the horizontal velocity components shown in Figs. 10a and 10b. This calculation requires the adjoint of an operator that inverts a distribution of barotropic relative vorticity to yield the components of the nondivergent (rotational) wind ($\nabla_o$). The adjoint of this operator will output the gradient of $R_1$ with respect to relative vorticity, given a distribution of the gradients of $R_1$ with respect to the nondivergent wind. It can be shown that the sensitivity to the nondivergent wind is equivalent to the sensitivity to the full wind [see (A1) in the appendix], and as a result, the sensitivities with respect to the full wind may be used in calculating the sensitivities with respect to the relative vorticity.

When the procedure outlined above is applied to the sensitivities with respect to the 700-hPa horizontal wind components, we deduce a distribution of relative vorticity sensitivity that is consistent with the previously described sensitivity vectors shown in Fig. 10c. The sensitivity with respect to relative vorticity is negative (positive) where the sensitivity vectors imply the addition of an anticyclonic (cyclonic) circulation to increase the value of the response function (Fig. 10c). It is apparent that the spatial scale of the distribution of relative vorticity sensitivity is much larger than that of the...
sensitivities to the horizontal wind components. This disparity in scales may be understood by considering the equation for the sensitivities of some arbitrary response function $R$ with respect to the vertical component of relative vorticity (also derived in the appendix): \begin{equation} \nabla^2 \left( \frac{\partial R}{\partial \zeta} \right) = - \left[ \frac{\partial}{\partial x} \left( \frac{\partial R}{\partial v} \right) - \frac{\partial}{\partial y} \left( \frac{\partial R}{\partial u} \right) \right]. \tag{7} \end{equation}

Mathematically, because this is an elliptic equation, we expect the “response” (the sensitivity to relative vorticity) to be of larger scale than the “forcing” (the spatial gradients of the sensitivities to the horizontal wind components). Physically, we recall that a localized vorticity perturbation has a broader influence compared with its spatial scale (the so-called action at a distance principle). Consequently, a given vorticity perturbation, remote from the maxima in sensitivities to the horizontal wind, can still have an impact on the response function.

As discussed earlier, horizontal temperature advection attributed to vorticity perturbations in the lower troposphere has a dominant effect on generating perturbations upstream of the box defining $R_1$. For vorticity perturbations in the upper troposphere, particularly for short forecast lead times, horizontal advection plays a less dominant role in generating upstream tempera-
Divergent circulations is associated with adiabatic warming at 500 hPa (denoted by gray, downward arrow) accompanying these (gent) circulations (denoted by gray, horizontal arrows). The subvorticity are accompanied by the generation of divergent (convergent) circulations (denoted by gray, horizontal arrows). The sub-sidence (denoted by gray, downward arrow) accompanying these divergent circulations is associated with adiabatic warming at σ = 0.85. As in Fig. 9, this temperature perturbation is subsequently advected into the region defining the response function.

**Fig. 13.** Schematic showing interpretation of sensitivities of average temperature to relative vorticity. Increases (decreases) in relative vorticity in regions within which the sensitivity to average temperature is positive (negative) result in increases to the temperature below (above) the region in which the vorticity is perturbed (sign of vorticity and sensitivity to vorticity indicated by “–” and “–” signs.) Dynamically, these increases (decreases) in vorticity are accompanied by the generation of divergent (convergent) circulations (denoted by gray, horizontal arrows). The sub-sidence (denoted by gray, downward arrow) accompanying these divergent circulations is associated with adiabatic warming at σ = 0.85. As in Fig. 9, this temperature perturbation is subsequently advected into the region defining the response function.

...ture perturbations, and vertical advection must be more important. To investigate this possibility, we consider the sensitivity of R1 to the τ = 30 h forecast of 500-hPa vorticity (Fig. 12a). From the distribution of this sensitivity gradient, which is nearly centered over the box defining R1, it is seen that a decrease in the forecast value of midtropospheric relative vorticity results in an increase of R1. Physically, a decrease in the cyclonic vorticity at 500 hPa will be associated with a negative horizontal divergence tendency. In a stably stratified atmosphere, the convergent flow at 500 hPa would be accompanied by adiabatic descent below 500 hPa and a concomitant warming of the atmospheric column. Thus created, this warm thermal perturbation would then be advected by the prevailing northwesterly flow into the box defining R1. The vertical dipole seen in the cross section through the sensitivity maximum (Fig. 12b) suggests that an increase in the lower tropospheric vorticity would also be accompanied by divergence and subsidence—leading to adiabatic warming in the lower troposphere. A conceptual model of this interpretation is shown in Fig. 13.

b. **Sensitivities with respect to geopotential**

The distribution of the sensitivity with respect to the derived quantity geopotential is computed using the adjoint of the calculation of the geostrophic wind. This sensitivity is qualitatively similar to the sensitivity with respect to vorticity, but of opposite sign and smaller scale because the (geostrophic) relative vorticity is proportional to the Laplacian of the geopotential. Where there is a local maximum (minimum) in sensitivity to relative vorticity, there is in general a local minimum (maximum) in sensitivity to geopotential (Figs. 10c,d). To the extent the perturbations are geostrophically balanced, this is consistent with adding a positive (negative) geopotential perturbation in the same region in which you would add a negative (positive) relative vorticity perturbation in order to increase the value of the response function.

c. **Summary of physical processes influencing R1**

The physical processes that determine the value of R1 are summarized in Fig. 14. The zero contour of the relative vorticity sensitivity nearly coincides with the axis of maximum sensitivity with respect to temperature, as well as being oriented nearly perpendicular to the basic state isotherms. This configuration highlights the importance of the generation of temperature perturbations by horizontal advection in a region extending from Minnesota into Manitoba in order to change R1. Since the sensitivities with respect to vertical motion and perturbation pressure are similar to the temperature sensitivities, all of the sensitivities illustrate the importance of temperature advection for the modification of R1.

7. **Other response functions**

In this section, two additional response functions are considered to demonstrate how physical insights into particular forecast aspects can be obtained through the use of an adjoint model and to further illustrate the dependency of forecast sensitivity gradients on the choice of response function. The response functions considered are the 36-h forecast average north–south temperature difference between two zonally oriented rows of grid points just north of Wisconsin and the 36-h forecast of the average zonal component of the wind over the same domain defined for R1. These two response functions, both defined on the σ = 0.85 surface, will be referred to as R2 and R3, respectively.

a. **Interpretation of sensitivity for R2**

The only nonzero component of the gradient of R2 with respect to the model forecast state (the gradient with respect to the 36-h forecast of temperature) is shown in Fig. 15a. For brevity, we consider the sensitivities with respect to the 30-h forecast of temperature, wind, and vertical velocity. Consistent with the interpretation for R1, adding temperature perturbations to increase (decrease) the temperature gradient at τ = 30 h where there is a dipole in temperature sensitivity (Fig. 15b), would lead to a change in R2. However, the generation of temperature perturbations via horizontal ad-
vection is somewhat more complicated, as either positive or negative temperature perturbations must be generated to the south or north of the baroclinic zone respectively to increase the forecasted north–south temperature difference. The sensitivity vectors suggest that adding a confluent flow along the baroclinic zone in northern Wisconsin and the upper peninsula of Michigan would increase the forecasted temperature gradient, consistent with horizontal frontogenesis. Likewise, a vertical cross section of sensitivity with respect to vertical motion, $w$, (Fig. 16) suggests that adding a thermally indirect circulation along the baroclinic zone will lead to an increase in the forecasted temperature gradient.

**b. Interpretation of sensitivity for $R_3$**

An altogether more complicated interpretation applies to the third response function, $R_3$. To change the value of $R_3$ by merely adding a zonal wind perturbation upstream of the domain in which the response function is defined and allowing that perturbation to subsequently be advected into the domain is not possible because any perturbation to the wind field will experience a rightward Coriolis torque. The sensitivity vectors for this response function appear to exhibit a behavior similar to an inertial oscillation. Consider the time evolution of the sensitivity vector in central Minnesota, which suggests adding a southeasterly wind at $\tau = 12$ h to increase $R_3$, but adding southwesterly wind at $\tau = 18$ h will also increase the value of $R_3$ (Fig. 17). The sensitivity vectors rotate clockwise by approximately $120^\circ$–$150^\circ$ every 6 h, consistent with an inertial oscillation at a mean latitude of $45^\circ$N. The sensitivities of $R_3$ with respect to temperature and pressure do not lend themselves to simple interpretation because the evolution of the zonal momentum is simultaneously influenced by not only the horizontal Coriolis force, but also horizontal pressure gradients and friction.

The results of this and the previous section suggest that for response functions associated with variables that are quasi-conserved (i.e., those variables for which advective dynamics are dominant), the synoptic and dynamical interpretations of the forecast sensitivities associated with those response functions are relatively straightforward. For those response functions associated with variables that are not conserved, the interpretation of the associated sensitivity gradients can be rather complicated.

**8. Summary and conclusions**

Atmospheric NWP models have become widely used tools in synoptic cases studies because they provide four-dimensionally consistent atmospheric states upon which diagnostic calculations may be performed. In addition, these models have become used as tools for an-
swering “what if?” questions: “How would a numerical forecast change if the PV associated with an upper trough (Fehlmann and Davies 1997) or tropical cyclone (McTaggart-Cowan et al. 2004) were removed from or altered in the initial model state?” Because adjoint models are efficient tools for evaluating the sensitivity of any differentiable function of a model state to previous forecast states, they, rather than NWP forecast models, are the most appropriate tools for answering these questions. Furthermore, an impact study can be more carefully formulated using information contained in adjoint-derived sensitivity gradients. For example, regions and variables that are determined to have little or no impact on the response function by the adjoint method need not be considered in an impact study (Fig. 18).

Despite the benefits of using adjoint model output, relatively few synoptic case studies have made use of adjoint-derived forecast sensitivities, either directly as a diagnostic, or to enhance an impact study. We believe that a reason for the lack of synoptic case studies employing adjoint output is that few synoptic and dynamical interpretations of the adjoint fields exist. As an initial step toward applying adjoint-derived forecast sen-

![Fig. 15. Temperature (light solid, interval 3 K) and sensitivity of $R_1$ with respect to temperature (dark, interval $2 \times 10^{-2}$ K K$^{-1}$, negative dashed with zero line omitted) at $\sigma = 0.85$ for (a) 36-h and (b) 30-h forecast initialized at 1200 UTC 10 Apr 2003. Also plotted in (b) are 30-h sensitivity vectors at $\sigma = 0.85$ (K m$^{-1}$ s) for $R_2$. Line G–H in (b) denotes orientation for cross section in Fig. 16.](image)

![Fig. 16. Cross section of potential temperature (light dashed, interval 3 K) and sensitivity of $R_2$ with respect to temperature (dark, interval $3 \times 10^{-2}$ K m$^{-1}$ s, negative contours dashed with zero line omitted) valid for 30-h forecast initialized at 1200 UTC 10 Apr 2003. Cross section orientation denoted by line in Fig. 15b.](image)

sitivity diagnostics for synoptic case studies, we explore the structure and evolution of adjoint sensitivity gradients is required. We suggest that for response functions of fundamental model variables (like temperature and wind), relatively straightforward interpretations might be afforded. In this study, we employ the adjoint method to calculate the sensitivities of response functions associated with the forecasted wind and temperature fields with the goal of providing synoptic interpretations of the corresponding forecast sensitivity gradients. Using the MM5 Adjoint Modeling System, forecast sensitivities for a 36-h forecast were calculated for three response functions: a forecast of lower tropospheric average temperature ($R_1$), temperature gradient ($R_2$), and average zonal wind ($R_3$). Novel aspects of this work include an interpretation of sensitivity with respect to the model forecast trajectory, a proposed means of visualizing the sensitivities of response functions with respect to the wind field through the construction of sensitivity vectors, and an interpretation of sensitivity with respect to derived variables.

All of the perturbations to the model initial and forecast state as implied by the distributions of forecast sensitivity gradient for $R_1$ can be directly linked to the addition or generation of temperature perturbations in upstream regions nearly coincident with the regions of maximum temperature sensitivity. For example, if a goal were to increase $R_1$, positive temperature perturbations could be added upstream directly. On the other hand, $R_1$ could also be increased by generating warm temperature perturbations through horizontal or vertical advection by the addition of appropriate wind perturbations relative to the basic state temperature field.

At any given forecast time preceding the time that $R_1$ is defined, all of the sensitivity gradients are geographically isolated in a region upstream from the box defin-
The sensitivity gradients propagate toward the region used to define the response function at approximately the advective speed of the basic state forecast. Additionally, using trajectory analysis, it was demonstrated that the processes controlling this response function were largely adiabatic, as the largest sensitivities stayed on approximately the same isentropic surface throughout the forecast period. While we have presented the results for a single case, our two year evaluation of the structure and evolution of the $R_1$ sensitivities in real-time suggests that not only is this case typical, but also that our interpretations of these particular forecast sensitivities are robust.

The distributions of sensitivity for the other response functions studied provide further physical insight into different aspects of this particular forecast. The sensitivity gradients for $R_2$ are consistent with the frontogenetical processes that are well known to contribute to an increase in a horizontal temperature gradient. The sensitivity vectors for this response function show that the addition of horizontal, confluent wind perturbations at $\tau = 30$ h with an axis of dilatation parallel to the isotherms leads to an increase in $R_2$. Likewise, the sensitivity with respect to vertical motion identifies the frontogenetical importance of adding a thermally indirect circulation to also increase $R_2$. Last, $R_3$ was shown to be sensitive to wind perturbations that are strongly influenced by the Coriolis force, as the sensitivity vectors undergo an evolution similar to an inertial oscillation.

The results described suggest that the clearest interpretation of forecast sensitivities may be found for response functions defined for nearly conserved variables (contrast the interpretations for $R_1$ and $R_3$). In particular, we believe significant synoptic insight to the dynamics and predictability of quasi-balanced weather systems can be gained through the evaluation of sensitivities of a response function defined as the PV at a point or averaged over a volume with respect to the PV of the model state. Sensitivities such as these should prove useful in identifying what parts of an initial or forecast state PV distribution should be changed to influence a specific aspect of a forecast.

Just as we have been able to combine the sensitivities with respect to the horizontal wind components into the sensitivity with respect to relative vorticity, Arbogast (1998) argues that for quasi-balanced flows, calculating the gradients with respect to the PV distribution can be an extremely useful aid in interpretation as the gradients with respect to the mass and wind fields are combined into a single gradient. Calculating sensitivities for other response functions, for other cases, and with respect to other derived variables should prove useful in

![Figure 17](image_url)
our understanding the structure and dynamics of forecast sensitivities.

The results of this study suggest immediate, practical applications to operational forecasting and to postforecast diagnosis and analysis. In real-time, forecasters concerned with a specific forecast aspect could compute sensitivities to that aspect for the entire forecast trajectory (rather than with respect to the model initial conditions). As a new analysis arrives, comparisons of the differences between say, the analysis and the model forecast of wind valid at that time, with the forecast sensitivity vectors for wind, would allow forecasters to assess the significance of differences between the analysis and model state on that forecast aspect at a later time. Last, the sensitivities to that forecast trajectory might also be useful in determining regions for targeting observations at later forecast times to improve that forecast aspect.

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APPENDIX

Calculation of Sensitivities with Respect to Derived Variables

a. Sensitivity with respect to nondivergent and irrotational winds

In section 6, it was stated that gradients of a response function with respect to the full wind were equivalent to the gradients of that same response function with respect to the nondivergent wind. To verify this relationship, consider a Helmholtz partitioning of the horizontal wind \( \mathbf{V} \) into its nondivergent \( \mathbf{V}_\phi \) and irrotational \( \mathbf{V}_\lambda \) components:

\[
\mathbf{V}_\phi + \mathbf{V}_\lambda = (1 \ 1) \begin{pmatrix} \mathbf{V}_\phi \\ \mathbf{V}_\lambda \end{pmatrix} = \mathbf{V}.
\]

Therefore, the adjoint operator needed to calculate the gradient with respect to the rotational component of the wind from the gradient with respect to the full wind is simply the transpose of the \( 1 \times 2 \) matrix \( (1 \ 1) \):

\[
\begin{pmatrix}
\frac{\partial R}{\partial \mathbf{V}_\phi} \\
\frac{\partial R}{\partial \mathbf{V}_\lambda}
\end{pmatrix} = (1 \ 1)^T \begin{pmatrix}
\frac{\partial R}{\partial \mathbf{V}}
\end{pmatrix}.
\]

(Fig. 18. Same as in Fig. 4d, except for location of initial temperature perturbation (denoted by the dot) at \( \sigma = 0.75 \) at the initial time. The box denotes location of temperature perturbation for original linearity test (Figs. 3 and 4). Note that the impact of the evolved perturbation is near zero over Wisconsin, despite the close proximity of the initial perturbations in the two experiments.)
b. Sensitivity with respect to vorticity and divergence

Rather than writing the adjoint of the relative vorticity inversion operator, there is an alternative way to derive the sensitivities of a response function with respect to vorticity (and divergence) given the sensitivities of that response function with respect to either the horizontal components of the wind or the streamfunction (\( \phi \)) and velocity potential (\( \chi \)). Following a procedure described by Arbogast (1998), we begin with the (linearized) expressions for vorticity and divergence in matrix form:

\[
\begin{pmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \chi} \\
\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \chi} \\
\frac{\partial}{\partial \phi} & \frac{\partial}{\partial \chi} & \xi' & \delta' \\
\frac{\partial}{\partial \chi} & \frac{\partial}{\partial \phi} & \xi' & \delta'
\end{pmatrix}
\]

Using the definition of the adjoint of the above\(^{A1}\) operator gives a coupled set of equations for the sensitivities of \( R \) with respect to vorticity and divergence:

\[
\begin{pmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial x} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \chi} \\
\frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \chi} \\
\frac{\partial}{\partial \phi} & \frac{\partial}{\partial \chi} & \xi' & \delta' \\
\frac{\partial}{\partial \chi} & \frac{\partial}{\partial \phi} & \xi' & \delta'
\end{pmatrix}
\begin{pmatrix}
\frac{\partial R}{\partial y} \\
\frac{\partial R}{\partial x} \\
\frac{\partial R}{\partial \phi} \\
\frac{\partial R}{\partial \chi}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial R}{\partial u} \\
\frac{\partial R}{\partial v} \\
\frac{\partial R}{\partial \theta} \\
\frac{\partial R}{\partial \phi}
\end{pmatrix}
\]

The equations for the sensitivities of \( R \) with respect to the vertical component of relative vorticity and the horizontal divergence can then be derived:

\[
\nabla^2 \left( \frac{\partial R}{\partial \phi} \right) = -\left( \frac{\partial}{\partial x} \frac{\partial R}{\partial v} - \frac{\partial}{\partial y} \frac{\partial R}{\partial u} \right) = \frac{\partial R}{\partial \phi}
\]

\[
\nabla^2 \left( \frac{\partial R}{\partial \chi} \right) = -\left( \frac{\partial}{\partial x} \frac{\partial R}{\partial \theta} + \frac{\partial}{\partial y} \frac{\partial R}{\partial \theta} \right) = \frac{\partial R}{\partial \chi}
\]

The final equalities in the above two equations follow because the Laplacian is a self-adjoint operator.

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\(^{A1}\)Note that with the assumption of homogeneous boundary conditions, the adjoint of \( \frac{\partial}{\partial x} \) is \( -\frac{\partial}{\partial x} \).