Vorticity Equation (AOSC470/600, Prof. Kleist)

Starting with the notion that we have an expression for the relative vorticity:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

We need to take the x derivative of the meridional momentum equation and y derivative of the zonal momentum equation (Martin 5.31a and 5.31b) to try and get terms that look like relative vorticity, i.e.:

$$\frac{\partial}{\partial x} \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + fu = -\frac{1}{\rho \frac{\partial p}{\partial y}} \right)$$

$$-\frac{\partial}{\partial y} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} - f v = -\frac{1}{\rho \frac{\partial p}{\partial x}} \right)$$

To compute the derivatives, all of the terms (except d/dt in each) require the use of the product rule for differentiation (quotient rule for last term of each expression). Doing so results in the following expanded derivatives of the momentum equations:

$$\begin{bmatrix}
\frac{\partial}{\partial x} \frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} + w \frac{\partial^2 v}{\partial x \partial z} + f \frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} + u \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} + w \frac{\partial^2 u}{\partial y \partial z} - f \frac{\partial v}{\partial y}
\end{bmatrix}$$

We can eliminate the term that involves a zonal derivative of Coriolis, and then combine the two equations into a single expression (boxes will now be used in grouping terms together):

$$\begin{bmatrix}
\frac{\partial}{\partial x} \frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial w}{\partial x} \frac{\partial v}{\partial z} + w \frac{\partial^2 v}{\partial x \partial z} + f \frac{\partial u}{\partial x} \\
\frac{\partial u}{\partial y} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial u}{\partial x} + u \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial y} \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial w}{\partial y} \frac{\partial u}{\partial z} + w \frac{\partial^2 u}{\partial y \partial z} - f \frac{\partial v}{\partial y}
\end{bmatrix}$$

$$= -\frac{1}{\rho} \frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial x} + \frac{1}{\rho} \frac{\partial^2 p}{\partial y \partial x} - \frac{1}{\rho} \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} + \frac{1}{\rho} \frac{\partial^2 p}{\partial y \partial x} - \frac{1}{\rho} \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x}$$
We can eliminate the second derivative pressure terms on the RHS (red boxes above). We will also combine the Eulerian u/v tendency terms (and switch the order of differentiation, green boxes) and combine the u, v, and w terms (purple) to get:

\[
\begin{split}
\frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + u \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial v}{\partial y} \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial z} - \frac{\partial^2 u}{\partial y \partial z} \\
= \frac{1}{\rho^2} \left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right)
\end{split}
\]

Next, pull out the d/dt in the first term, d/dx in the second term, d/dy in the third term, and d/dz in the fourth term, and bring out the Coriolis parameter in the term in the second row (blue):

\[
\begin{split}
\frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + u \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + v \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + w \frac{\partial}{\partial z} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \\
= \frac{1}{\rho^2} \left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right)
\end{split}
\]

For the first four terms, we will replace (dv/dx-du/dy) with zeta/relative vorticity, and rewrite the first term of the second row in terms of relative vorticity times divergence:

\[
\begin{split}
\frac{\partial}{\partial t} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial x} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \\
= \frac{1}{\rho^2} \left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right)
\end{split}
\]

To get to:

\[
\begin{split}
\frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} + \frac{\partial}{\partial x} \left( \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial y} \right) + \frac{\partial}{\partial z} \left( \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial y} \right) + f \frac{\partial \zeta}{\partial x} + \frac{\partial \zeta}{\partial y} + v \frac{\partial \zeta}{\partial y} + w \frac{\partial \zeta}{\partial z} \\
= \frac{1}{\rho^2} \left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial \rho}{\partial y} \right)
\end{split}
\]

We can define the Lagrangian derivative of Coriolis as:

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z}
\]

But we can take advantage of the fact that the only non-zero term is the term that contains the meridional derivative and rewrite this as:
\( \frac{df}{dt} = \nu \frac{\partial f}{\partial y} \)

We combine the Eulerian and advective derivatives for relative vorticity (green and purple) into the total derivative, and replace the \( v(df/dy) \) term, we can condense the previous expression as:

\[
\frac{d\zeta}{dt} + \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \left( \frac{\partial w}{\partial z} - \frac{\partial u}{\partial z} \right) + f \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \frac{df}{dt} = \frac{1}{\rho^2} \left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial x} \frac{\partial \rho}{\partial y} \right)
\]

Combining the relative vorticity and Coriolis into a single total derivative and combining the terms that contain a multiplication of the divergence:

\[
\frac{d(\zeta + f)}{dt} + (\zeta + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left( \frac{\partial w}{\partial z} - \frac{\partial u}{\partial z} \right) = \frac{1}{\rho^2} \left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial x} \frac{\partial \rho}{\partial y} \right)
\]

Finally, we move everything to the RHS to come up with the vorticity equation!

\[
\frac{d(\zeta + f)}{dt} = -(\zeta + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left( \frac{\partial w}{\partial z} - \frac{\partial u}{\partial z} \right) + \frac{1}{\rho^2} \left( \frac{\partial p}{\partial y} \frac{\partial \rho}{\partial x} - \frac{\partial \rho}{\partial x} \frac{\partial \rho}{\partial y} \right)
\]

This is the vorticity equation in height coordinates (Martin 5.33). This equation states that the Lagrangian tendency (time rate of change following the flow) of the absolute vorticity consists of a divergence term (orange), tilting term (green), and solenoid term (yellow). See Martin Figures 5.11, 5.12, and 5.13 for schematic examples.

This same procedure can be repeated to derive a vorticity equation in isobaric (pressure) coordinates. Starting with:

\[
\begin{align*}
\frac{\partial}{\partial x} \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial \rho} + fu & = -\frac{\partial \phi}{\partial y} \\
- \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial \rho} - fv & = -\frac{\partial \phi}{\partial x} \right) \end{align*}
\]

Which eventually leads to:

\[
\frac{d(\zeta + f)}{dt} = -(\zeta + f) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + \left( \frac{\partial \omega}{\partial x} \frac{\partial \rho}{\partial y} - \frac{\partial \omega}{\partial y} \frac{\partial \rho}{\partial x} \right)
\]

In isobaric coordinates, we lose the solenoidal term. We can finally rewrite this as the isobaric vorticity equation:

\[
\frac{\partial \zeta}{\partial t} = -\vec{V} \cdot \nabla (\zeta + f) - \omega \frac{\partial \zeta}{\partial \rho} \left( \zeta + f \right) \left( \nabla \cdot \vec{V} \right) + \kappa \cdot \left( \frac{\partial \vec{V}}{\partial \rho} \times \nabla \omega \right)
\]